Unlikely Intersections and applications to Diophantine Geometry

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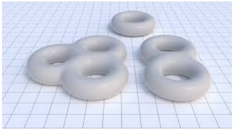
- Find all the solutions.

Diophantine Geometry



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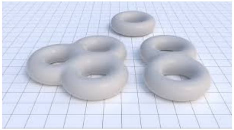

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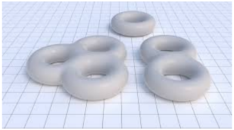

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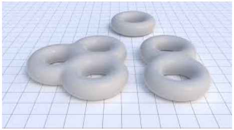

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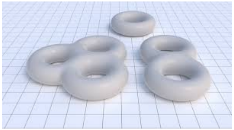

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"geometry determines arithmetic"

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This strategy inspired the formulation of several conjectures, as

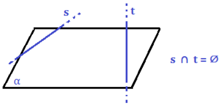
- Manin-Mumford conjecture (Laurent, Raynaud, Hindry, Hurshowski, Szpiro-Ullmo-Zhang, Pila-Zannier),
- Mordell-Lang conjecture (Laurent, Faltings, Hindry, Vojta, McQuillan),

recently generalized by Zilber, Bombieri-Masser-Zannier in the case of tori and more generally by Pink in the setting of mixed Shimura varieties.

Unlikely Intersections - General philosophy

$X, Y \subset T$ subvarieties of an ambient variety of dimension n.

General expectation:
$\dim(X \cap Y) \leq \dim X + \dim Y - n$; in particular
$X \cap Y = \varnothing$ if $\dim X + \dim Y < n$
unless there is some *geometric reason* for this.



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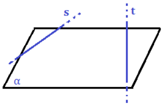

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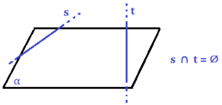

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More generally, let us fix $X \subset T$ and let us consider $\mathcal{Y}$ a family of subvarieties of T of codimension $> \dim X$.

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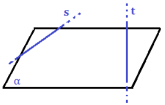

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More generally, let us fix $X \subset T$ and let us consider $\mathcal{Y}$ a family of subvarieties of T of codimension $> \dim X$.

General expectation: $X \cap Y = \emptyset$ in "most of" the cases;
in particular, $\bigcup_{Y \in \mathcal{Y}} X \cap Y$ is "small" with respect to X.

Zilber-Pink Conjecture for semiabelian varieties

Conjecture (Zilber-Pink)

Let A be a complex semiabelian variety and let X be an irreducible subvariety of A of dimension d. Let

$$A^{[d+1]} := \bigcup_{codim\ H \geq d+1} H.$$

Then, if X is not contained in any proper algebraic subgroup of A, the intersection $X \cap A^{[d+1]}$ is not Zariski-dense in X.

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- few known cases – if X is a curve or $\dim X = n - 2$ (Maurin, Bombieri-Masser-Zannier, Viada, Rémond, Habegger-Pila, Barroero-Kühne-Schmidt ...).

Unlikely intersections in a torus

Theorem (Bombieri-Masser-Zannier 1999)

Let $\mathcal{C} \subset (\mathbb{C}^*)^n$ be an irreducible curve defined over $\overline{\mathbb{Q}}$ such that it is not contained in a translate of a proper algebraic subgroup of $(\mathbb{C}^*)^n$. Then, $\mathcal{C} \cap (\mathbb{C}^*)^{[2]}$ is finite.

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Example:

There exist only finitely many $x \in \mathbb{C}$ satisfying

$$\begin{cases} x^{a_1}(1-x)^{a_2}(1+x)^{a_3} = 1 \\ x^{b_1}(1-x)^{b_2}(1+x)^{b_3} = 1 \end{cases}$$

with $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{Z}^3$ linearly independent.

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Height of an algebraic number:
measure of the *"complexity"* of the number.
Ex: if $\alpha = \frac{a}{b} \in \mathbb{Q}$, then:

$$H(\alpha) = \max\{|a|, |b|\}.$$

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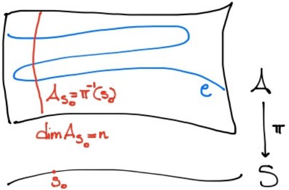
Analogous results for curves in families of abelian varieties (Barroero-C.)



Application to Pell equation in polinomials

Let $D \in \mathbb{C}[t]$ be a polynomial without multiple roots; we ask whether there exist $A, B \in \mathbb{C}[t]$ with $B \neq 0$ such that

$$A^2 - DB^2 = 1.$$

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Question: How many Pellian polynomials do we have?

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Results of unlikely intersections → solvability of

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Example:

Let $D_\lambda(t) = t^6 + t + \lambda$. There are only finitely many $\lambda_0 \in \mathbb{C}$ such that $t^6 + t + \lambda_0$ is Pellian.

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One can study the same questions for families of "generalized" Pell equations

$$A^2 - DB^2 = F$$

with $D, F \in \mathbb{C}[\lambda, t]$.

Application to Pell equation in polinomials

Theorem (Barroero-C. 2020)

Let $D_\lambda(t) \in \overline{\mathbb{Q}}(\lambda)[t]$ some "nice" polynomial. Let $F_\lambda(t) \in \overline{\mathbb{Q}}[t, \lambda] \setminus \{0\}$. Then, either the generalized Pell equation has an identical solution, or there exist at most finitely many $\lambda_0 \in \mathbb{C}$ such that the specialized equation

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Example:

Let $D_\lambda(t) = (t-\lambda)(t^7 - t^3 - 1)$ and $F(X) = 4t + 1$. Then there exist only finitely many $\lambda_0 \in \mathbb{C}$ such that the equation

$$A^2 - (t-\lambda_0)(t^7 - t^3 - 1)B^2 = 4t + 1$$

has a non trivial solution $A, B \in \mathbb{C}[t]$ with $B \neq 0$.