First passage percolation and competition on graphs

Elisabetta Candellero

Roma Tre University

Giornata INdAM – Roma Tre – May 2022

What I work on and why

I study the long-time behavior of some processes that spread/develop on graphs that are *structurally* very different from each other.

Why is it interesting?

1. Sometimes: possible to witness *fundamentally* different behavior of the process, according to what properties characterize the graph
2. If graphs mimic the structure of social networks... study spread of infections/misinformation (etc) in a society

Examples

Why looking only at processes on lattices and Euclidean structures?

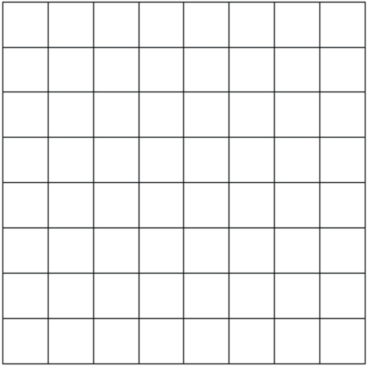


Figure: Two-dimensional grid (Euclidean).

Examples

Different Geometries ⇒ Great variety of new (exciting) questions!

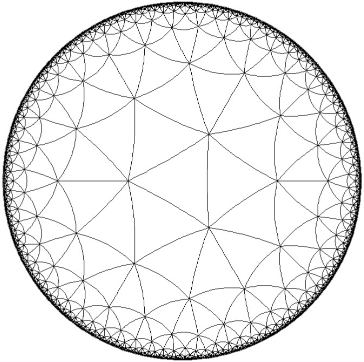
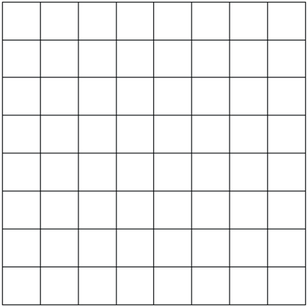


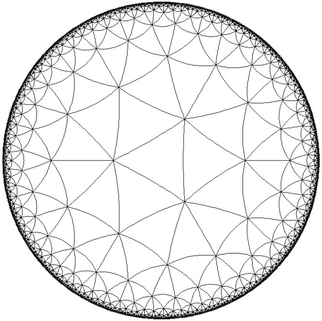
Figure: Tessellation of Hyperbolic Plane (Non Euclidean).

Examples

Start **infection** at a vertex and inductively **infect random neighbors** at constant rate. Is the long-time behavior the same in both settings?



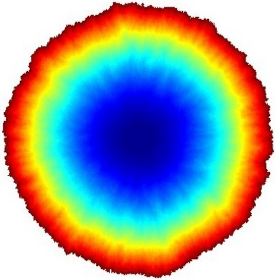
Lattice



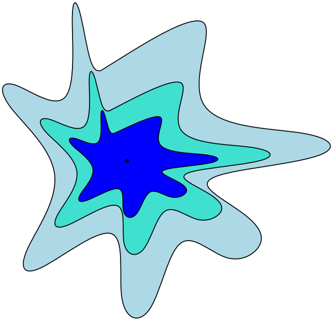
Hyperbolic graph

Examples

Fundamentally Different Behavior!



Lattice



Hyperbolic graph

Model for competition

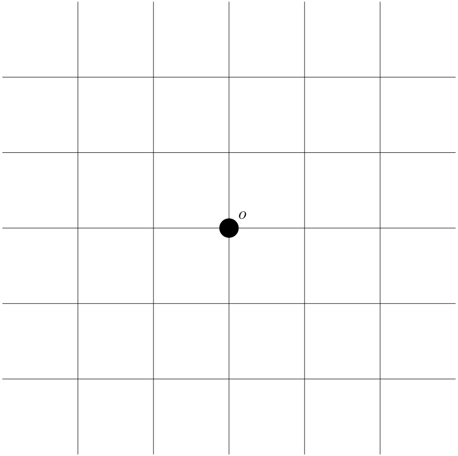
We deal with 2 First-passage percolation processes:

$$\mathbf{FPP}_1 \quad \text{and} \quad \mathbf{FPP}_\lambda$$

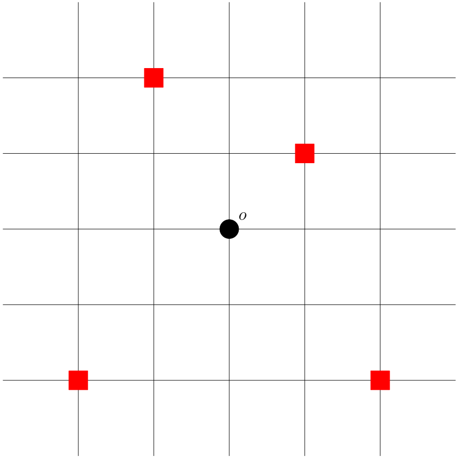
spreading on a graph G.
Choose 2 parameters: $\boldsymbol{\lambda} > 0$ and $\boldsymbol{\mu} \in (0, 1)$.

- At the origin o place a **black** particle;
- At every $x \in V(G) \setminus \{o\}$ place a **red** particle (call it **SEED**) with probability $\boldsymbol{\mu}$ independently for all vertices.

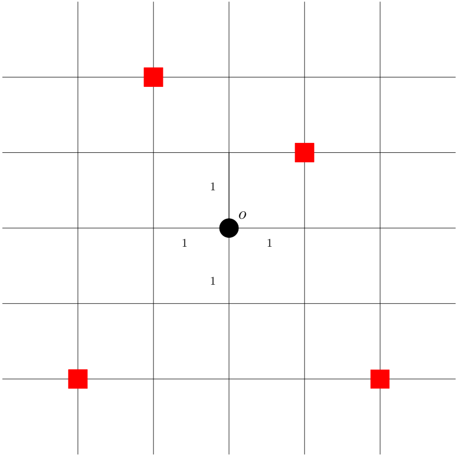
Model: Example on $\mathbb{Z}^2$



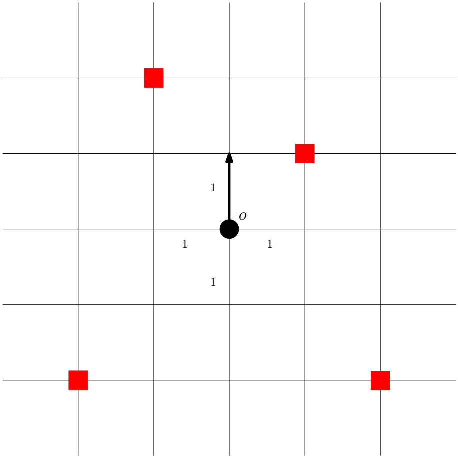
Model: Example on $\mathbb{Z}^2$



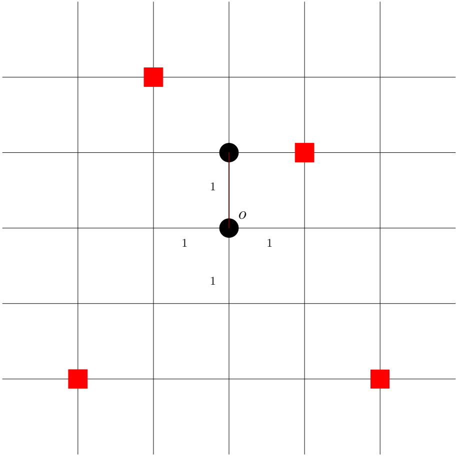
Dynamics: Example on $\mathbb{Z}^2$



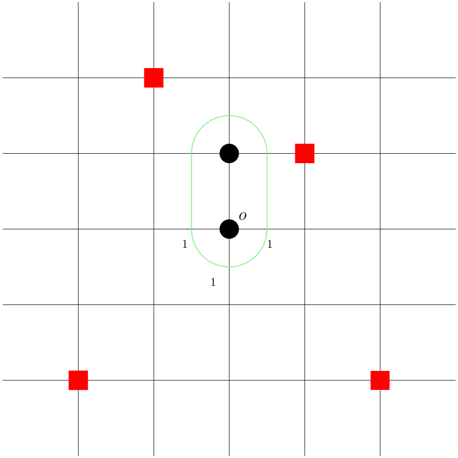
Dynamics: Example on $\mathbb{Z}^2$



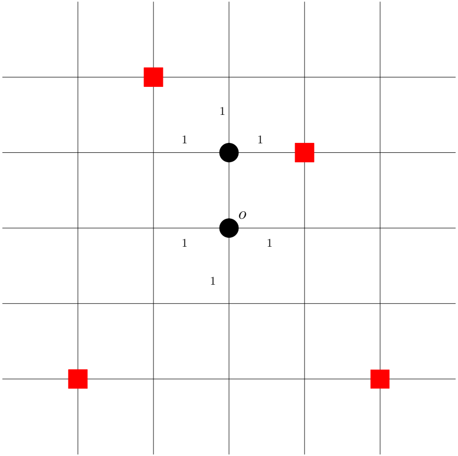
Dynamics: Example on $\mathbb{Z}^2$



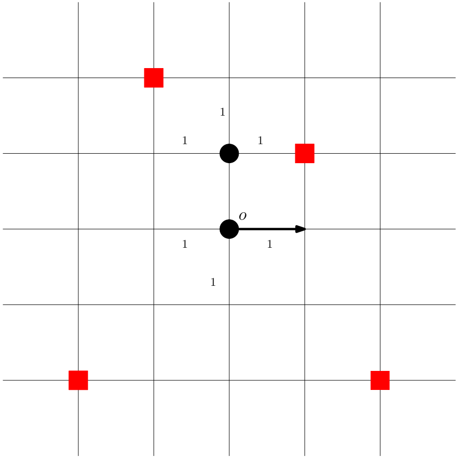
Dynamics: Example on $\mathbb{Z}^2$



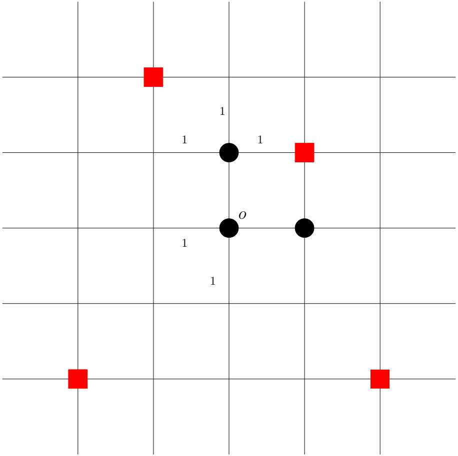
Dynamics: Example on $\mathbb{Z}^2$



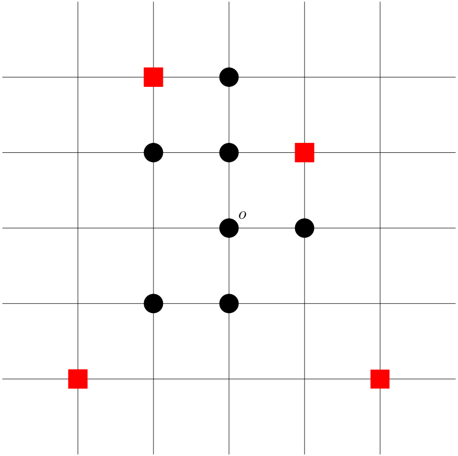
Dynamics: Example on $\mathbb{Z}^2$



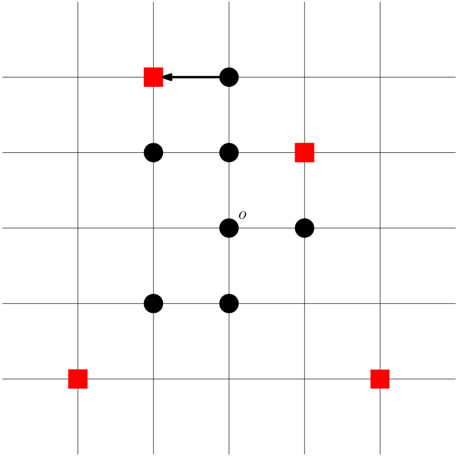
Dynamics: Example on $\mathbb{Z}^2$



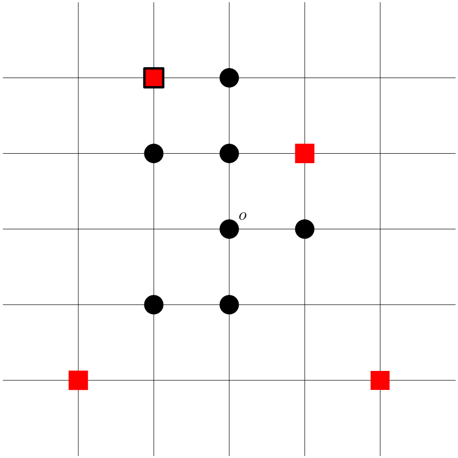
Dynamics: Example on $\mathbb{Z}^2$



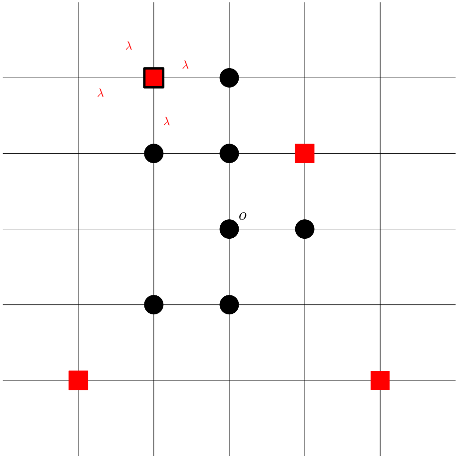
Dynamics: Example on $\mathbb{Z}^2$



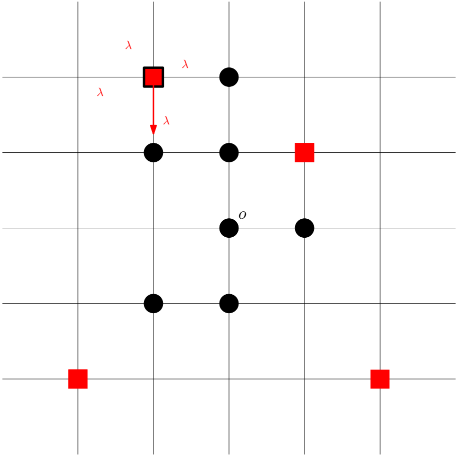
Dynamics: Example on $\mathbb{Z}^2$



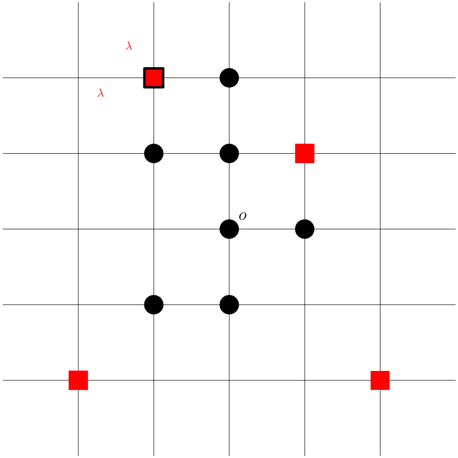
Dynamics: Example on $\mathbb{Z}^2$



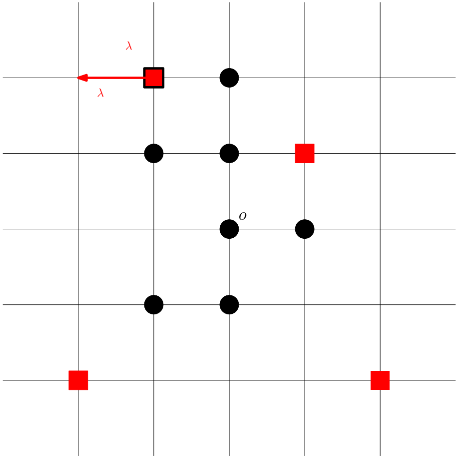
Dynamics: Example on $\mathbb{Z}^2$



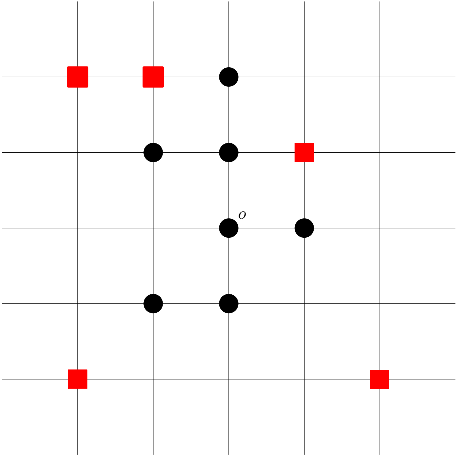
Dynamics: Example on $\mathbb{Z}^2$



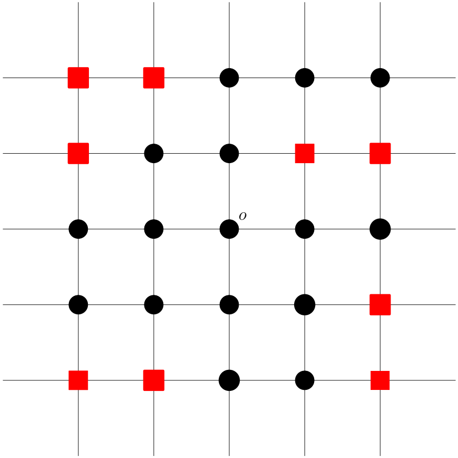
Dynamics: Example on $\mathbb{Z}^2$



Dynamics: Example on $\mathbb{Z}^2$



Dynamics: Example on $\mathbb{Z}^2$



Dynamics

FPP$_1$: starts at o and has passage times $\sim \mathrm{Exp}(1)$.

Inductively, place an exponential clock of rate 1 on all edges neighboring the current **black cluster** and look at which one rings first, then "grow" the black cluster in that direction.

In the meanwhile, **seeds are inactive (sleeping)**.

FPP$_\lambda$: when a **seed** is "*activated*", it starts spreading FPP at rate $\boldsymbol{\lambda} > 0$.

NOTE: Occupied vertices stay so for ever.

Model/Questions

We have these two processes (think of infections) that spread at different rates (**black** at rate $\mathbf{1}$ and **red** at rate $\boldsymbol{\lambda} > 0$) and are competing for space.

Some questions about the model:

- **Survival**: When either process occupies an INFINITE CONNECTED region of the graph
- **Coexistence**: When both processes survive simultaneously
- (**Monotonicity** –*if time allows*: Probability of $\mathbf{FPP}_1$ surviving is/isn't monotone in μ)

Related works

Model introduced by Sidoravicius and Stauffer (*Invent. Math.*) as

First-Passage Percolation in Hostile Environment

to understand MDLA on $\mathbb{Z}^d$, $d \geq 2$. (Coupling MDLA with FPPHE)

Theorem [Sidoravicius and Stauffer, 2019] On $\mathbb{Z}^d$ for $d \geq 2$, for all $\boldsymbol{\lambda} \in (0,1)$ if $\boldsymbol{\mu}$ is small enough:

$$\mathbb{P}_{\boldsymbol{\mu}}\left[\textbf{FPP}_1 \text{ survives and } \textbf{FPP}_\lambda \text{ does not}\right] > 0.$$

Related works

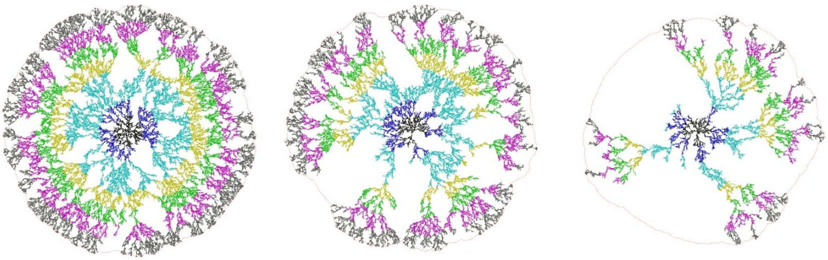


Figure: FPPHE on $\mathbb{Z}^2$ with $\lambda = 0.7$ and $\mu = 0.027, 0.029, 0.030$ respectively. Colors $\Rightarrow$ different epochs of the growth of $\mathbf{FPP}_1$. The whole white region within the thin boundary is occupied by activated $\mathbf{FPP}_\lambda$. (Picture by A.Stauffer)

Related works

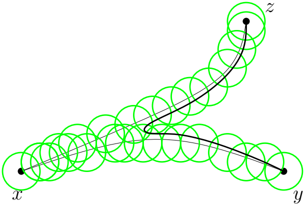
[Finn and Stauffer, 22+] Study coexistence of $\mathbf{FPP}_1$ and $\mathbf{FPP}_\lambda$ on $\mathbb{Z}^d$.

If λ is small enough, then there is a range of values for μ so that both processes can coexist with positive probability!

Our setting: hyperbolic and non-amenable graphs

Hyperbolic graphs

A graph is δ hyperbolic (for some $\delta \geq 0$) if for any triplet of vertices $\{x, y, z\}$:



Our setting: hyperbolic and non-amenable graphs

Non-amenable graphs

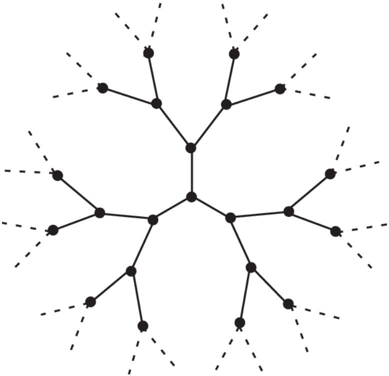
For all finite sets of vertices A, let

$$\partial A := \{x \in A \ : \ \exists\, y \notin A,\ \{x, y\} \in E(G)\}.$$

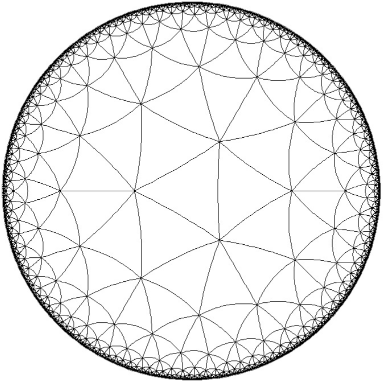
G is non-amenable if there is a constant $\mathbf{c} > 0$ such that

$$\inf_{|A|<\infty} \frac{|\partial A|}{|A|} \geq \mathbf{c}.$$

Hyperbolic non-amenable graphs: Two Examples



Infinite binary tree



Hyperbolic tessellation

Why do we care about such graphs?

FPP can be seen as a model for **spread of infection**, or the **spread of a false rumor within a network**. Think of the following:

Misinformation (as $\mathbf{FPP}_1$) starts from an individual in the network.

When *detected* by the detecting stations (**seeds**), they try to stop it.

Proven: there are models for real-world networks with intrinsic *hyperbolic* and (local) *non-amenable* properties.

Results on Survival and Coexistence

Theorem [C. and Stauffer, 2021] Let G be infinite, **hyperbolic**, **non-amenable**, vertex-transitive, bounded degree, then:

(i) For all $\lambda > 0$ and for all μ small enough,

$$\mathbb{P}_{\mu}[\textbf{FPP}_1 \text{ survives indefinitely}] > 0.$$

(ii) For all $\lambda > 0$, for all $\mu \in (0, 1)$ we have

$$\mathbb{P}_{\mu}[\textbf{FPP}_{\lambda} \text{ survives indefinitely}] = 1.$$

Corollary [C. and Stauffer, 2021] For G as above, for all λ, all μ small,

$$\mathbb{P}_{\mu}[\textbf{FPP}_1 \text{ and } \textbf{FPP}_{\lambda} \text{ coexist}] > 0.$$

Reasons

We use 2 facts:

(A) First-passage percolation **grows linearly in time**, with high probability.

(B) Gromov's result on δ-hyperbolic graphs: if a path joining two vertices moves away from geodesic $\Rightarrow$ it takes an **exponential** detour.

$(A) + (B) \Rightarrow$ FPP moving away from geodesics has **exponential** delays

If short of time

Monotonicity in μ?

It turns out that (percolation arguments)

(for all λ) if μ very close to 1 $\quad\Rightarrow\quad$ **FPP**$_1$ a.s. doesn't survive,

whereas in all known cases

at least if μ and λ small enough $\quad\Rightarrow\quad$ **FPP**$_1$ survives w.p.p.

Moreover, **seeds** get in the way of **FPP**$_1$ because they can "interrupt" **black** paths.

Monotonicity in μ? – Natural conjecture

Thus it is natural to conjecture that for (at least) **small λ**

$$\mathbb{P}_{\mu}(\mathbf{FPP}_1 \text{ survives}) \text{ is monotone in } \mu.$$

Monotonicity in μ? – Natural conjecture

Thus it is natural to conjecture that for (at least) **small** λ

$$\mathbb{P}_{\mu}(\mathbf{FPP}_1 \text{ survives}) \text{ is monotone in } \mu.$$

FALSE!

Results on Monotonicity

Theorem [C. and Stauffer, 2021+] There is an infinite (hyperbolic and non-amenable) graph G s.t. when λ is small enough, we can find two values $0 < \mu_1 < \mu_2 < 1$:

$$\mathbb{P}_{\mu_1}(\textbf{FPP}_1 \text{ survives}) = 0,$$

and

$$\mathbb{P}_{\mu_2}(\textbf{FPP}_1 \text{ survives}) > 0.$$

Results on Monotonicity

Theorem [C. and Stauffer, 2021+] There is an infinite (hyperbolic and non-amenable) graph G s.t. when λ is small enough, we can find two values $0 < \mu_1 < \mu_2 < 1$:

$$\mathbb{P}_{\mu_1}(\mathbf{FPP}_1 \text{ survives}) = 0,$$

and

$$\mathbb{P}_{\mu_2}(\mathbf{FPP}_1 \text{ survives}) > 0.$$

This means that adding **seeds** might actually *help* $\mathbf{FPP}_1$!

Social network

Next step:

Understanding how misinformation spreads in real-world networks

Very ambitious goal though… in fact we're still far off!

Social network

Graphs (vaguely) modeling social networks might look like this:

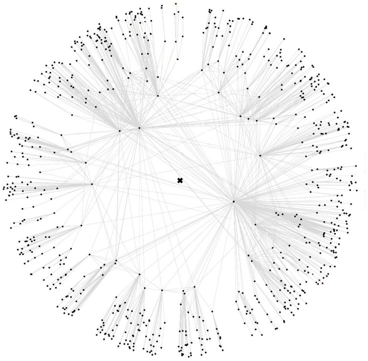


Figure: Random hyperbolic graphs present an intrinsic hyperbolic and non-amenable structure.

Social network

However, a *real*-world network looks like this (still a log way to go...):

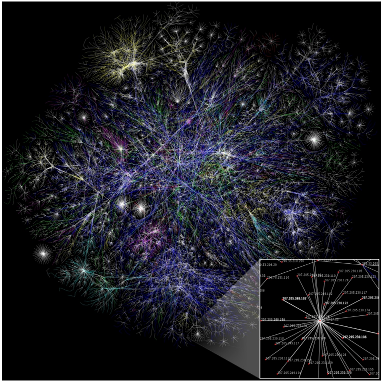


Figure: The internet – Picture by Wikipedia.

Thank you for your attention!