Modeling and Optimal Control of a Tentacle-like Soft-Robot

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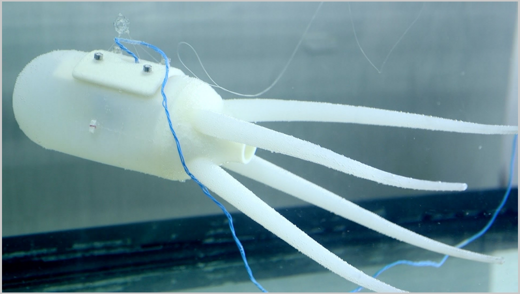
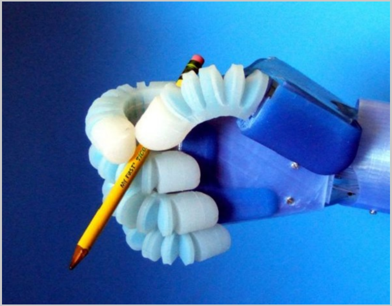
Motivations





Soft Robotics

- Soft materials, great adaptability
- Safe robot/human interactions
- Microsurgery
- Rehabilitation
- Exploration
- Rescue

...

Modeling

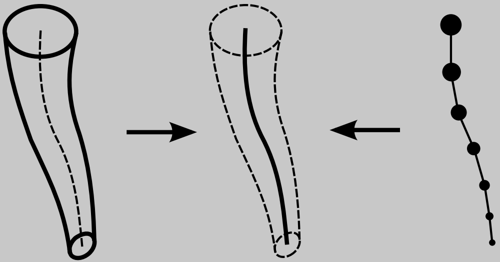
- Neglect torsion $\Longrightarrow$ planar 2D model
- Translating physical properties:

$$\text{Variable thickness} \Longrightarrow \text{non-uniform} \begin{cases} \text{mass distribution} \\ \text{resistance to bending} \\ \text{curvature constraints} \end{cases}$$

Bending $\gg$ Elongation $\Longrightarrow$ Inextensibility

Muscles activation $\Longrightarrow$ Control of local curvature

- Approximation via constrained non-uniform chain (multi-pendulum)



Constraints

Lagrangian description via a system of N linked particles

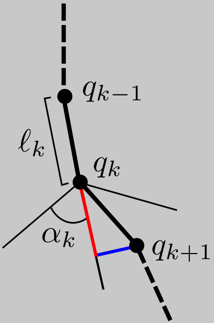
q_k: position of the k-th particle (joint)

m_k: mass of the k-th particle

ℓ_k: distance between joints $k-1$ and k

α_k: maximum angle between joints $k-1$, k and $k+1$

$$\sum_{k=1}^{N} m_k = 1, \quad \sum_{k=1}^{N} \ell_k = 1$$



Inextensibility: $|q_k - q_{k-1}| = \ell_k$

Curvature constraint: $(q_{k+1} - q_k) \cdot (q_k - q_{k-1}) \geq \ell_k^2 \cos(\alpha_k)$

Curvature control: $(q_{k+1} - q_k) \times (q_k - q_{k-1}) = \ell_k^2 \sin(\alpha_k u_k)$, $u_k \in [-1, 1]$

Bending constraint: $(q_{k+1} - q_k) \times (q_k - q_{k-1}) = 0$

Notation: $a \times b = a \cdot b^{\perp}, \quad b^{\perp} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} b$

Discrete to Continuous limit

Assume

$$\ell_k \equiv \ell = \frac{1}{N}, \quad m_k = \ell \rho_k, \quad \alpha_k = \ell \omega_k$$

and form the (suitably rescaled) Lagrangian

$$\mathcal{L}_N(t, q, \dot{q}) = \sum_{k=1}^{N} \begin{cases} \frac{1}{2}\ell\rho_k|\dot{q}_k|^2 \\ -\frac{1}{2\ell}\sigma_k\left(|q_k - q_{k-1}|^2 - \ell^2\right) \\ -\frac{1}{\ell^3}\nu_k\left(\cos(\ell\omega_k) - \frac{1}{\ell^2}(q_{k+1} - q_k)\cdot(q_k - q_{k-1})\right)_+^2 \\ -\frac{1}{2\ell}\mu_k\left(\sin(\ell\omega_k u_k) - \frac{1}{\ell^2}(q_{k+1} - q_k)\times(q_k - q_{k-1})\right)^2 \\ -\frac{1}{2\ell^5}\varepsilon_k\Big((q_{k+1} - q_k)\times(q_k - q_{k-1})\Big)^2 \end{cases}$$

where, for $k = 1, ..., N$,
σ_k is a Lagrange multiplier (tension),
$\nu_k, \mu_k, \varepsilon_k$ are penalty parameters (elastic constants, bending stiffness)

Discrete to Continuous limit

Consider regular (scalar or vector) functions $\chi(s,t)$ such that

$$\chi(k\ell,t)=\chi_k(t)\,,\qquad \text{for } \chi=\rho,\omega,q,\sigma,\nu,\mu,\varepsilon\,.$$

Using Taylor expansions and taking the limit as $\ell\to 0$ ($N\to\infty$) we get the continuous Lagrangian

$$\mathcal{L}(t,q,q_t)=\int_0^1\left\{\frac{1}{2}\rho|q_t|^2-\frac{1}{2}\sigma\left(|q_s|^2-1\right)-\frac{1}{4}\nu\left(|q_{ss}|^2-\omega^2\right)_+^2\right.$$

$$\left.-\frac{1}{2}\mu\left(\omega u-q_s\times q_{ss}\right)^2-\frac{1}{2}\varepsilon\left|q_{ss}\right|^2\right\}ds\,dt$$

Model parameters:

thin joints weigh less than thick joints $\implies$ $\rho\searrow$

thin joints bend easier than thick joints $\implies$ $\nu,\mu,\varepsilon\searrow$ and $\omega\nearrow$

Controlled Dynamics

Assuming boundary (free end) conditions and initial conditions

$$
(BC + IC) = \begin{cases} q(0,t) = 0 & t \in (0,T) \quad \text{anchor point} \\ q_s(0,t) = -e_2 & t \in (0,T) \quad \text{fixed tangent} \\ q(s,0) = q^0(s) & s \in (0,1) \quad \text{initial profile} \\ q_t(s,0) = q^1(s) & s \in (0,1) \quad \text{initial velocity} \end{cases}
$$

then a stationary point (q,σ) of the action $\mathcal{S} = \int_0^T \mathcal{L}(t,q,q_t)dt$ satisfies

$$
\begin{cases} \rho q_{tt} = \left[\sigma q_s - H q_{ss}^{\perp}\right]_s - \left[G q_{ss} + H q_s^{\perp}\right]_{ss} & \text{in } (0,1)\times(0,T) \\ |q_s|^2 = 1 \quad \text{in } (0,1)\times(0,T) & \text{inextensibility} \\ q_{ss}(1,t) = 0 \quad t \in (0,T) & \text{zero bending moment at } s=1 \\ q_{sss}(1,t) = 0 \quad t \in (0,T) & \text{zero shear stress at } s=1 \\ \sigma(1,t) = 0 \quad t \in (0,T) & \text{zero tension at } s=1 \end{cases}
$$

where

$$
G(q,\nu,\varepsilon,\omega) = \varepsilon + \nu\left(|q_{ss}|^2 - \omega^2\right)_+ \qquad H(q,\mu,u,\omega) = \mu\left(\omega u - q_s \times q_{ss}\right)
$$

Equilibria

Bending Moment VS Curvature Control

Let $u \in C^2([0,1])$ and define $\bar{\omega}(\mu,\omega,\varepsilon) := \dfrac{\mu\omega}{\mu+\varepsilon} \leq \omega$.

Then the stationary problem

$$\begin{cases} \left[\sigma q_s - H q_{ss}^{\perp}\right]_s - \left[G q_{ss} + H q_s^{\perp}\right]_{ss} = 0 & \text{in } (0,1) \\ |q_s|^2 = 1 & \text{in } [0,1] \\ q(0) = 0\,, \quad \sigma(1) = 0 & \\ q_s(0) = -e_2\,, \quad q_{ss}(1) = 0\,, \quad q_{sss}(1) = 0 & \end{cases}$$

admits a unique solution $(q,\sigma) \in C^4([0,1]) \times C^2([0,1])$ which is given by

$$\begin{cases} q_s \times q_{ss} = \bar{\omega} u & \text{in } (0,1) \\ \sigma = \varepsilon(\bar{\omega} u)^2 & \text{in } [0,1] \\ |q_s|^2 = 1 & \text{in } [0,1] \\ q(0) = 0\,, \quad q_s(0) = -e_2 & \\ q_{ss}(1) = 0\,, \quad q_{sss}(1) = 0 & \end{cases}$$

Reachability

Touch a point with the tentacle tip and minimum effort

Given $q^* \in \mathbb{R}^2$ and $\tau_0 > 0$

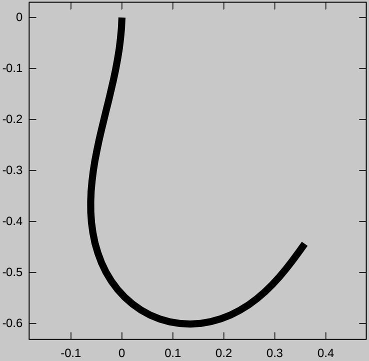
$$\text{Minimize} \quad \frac{1}{2}\int_0^1 u^2 ds + \frac{1}{2\tau_0}|q(1) - q^*|^2 \qquad \text{subject to} \begin{cases} q_s \times q_{ss} = \bar{\omega} u \\ |u| \leq 1 \\ |q_s|^2 = 1 \\ q(0) = 0 \\ q_s(0) = -e_2 \\ q_{ss}(1) = 0 \\ q_{sss}(1) = 0 \end{cases}$$

Using $|q_{ss}| = \bar{\omega}|u|$ to eliminate u yields

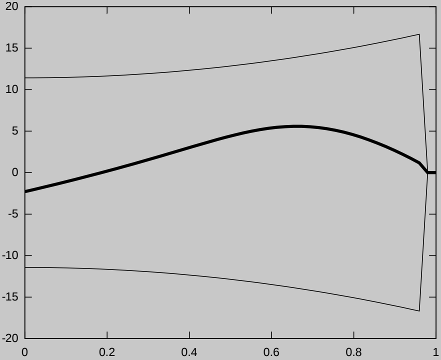
$$\text{Minimize} \quad \frac{1}{2}\int_0^1 \frac{1}{\bar{\omega}^2}|q_{ss}|^2 ds + \frac{1}{2\tau_0}|q(1) - q^*|^2 \qquad \text{subject to} \begin{cases} |q_{ss}| \leq \bar{\omega} \\ |q_s|^2 = 1 \\ q(0) = 0 \\ q_s(0) = -e_2 \\ q_{ss}(1) = 0 \\ q_{sss}(1) = 0 \end{cases}$$

Reachability

Target point $q^* = (0.35, -0.45)$



Shape

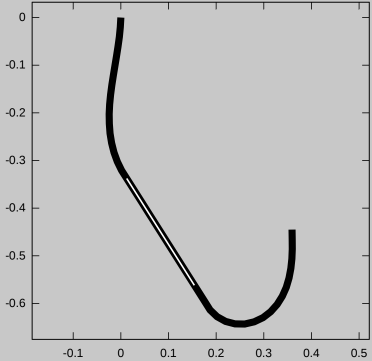


Signed Curvature

$$\bar{\omega}(s) = 4\pi(1 + s^2)$$

Reachability

Target point $q^* = (0.35, -0.45)$



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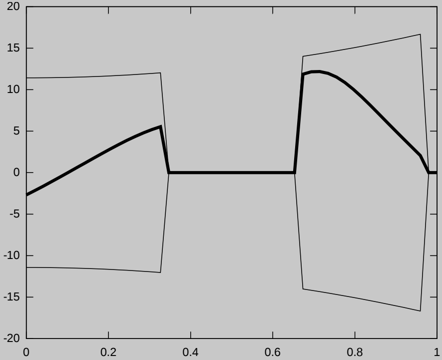


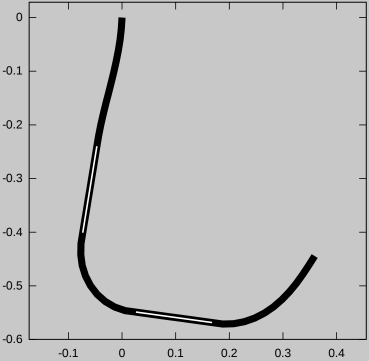
Signed Curvature

Mechanical breakdown

$$\bar{\omega}(s) = \begin{cases} 0 & s \in (0.35, 0.65) \\ 4\pi(1+s^2) & \text{otherwise} \end{cases}$$

Reachability

Target point $q^* = (0.35, -0.45)$



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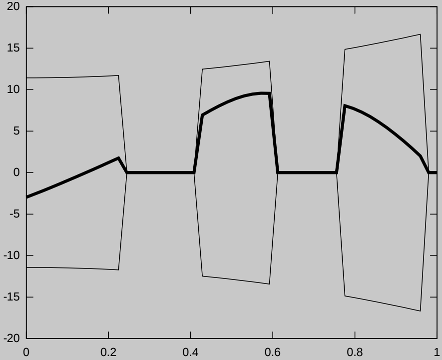


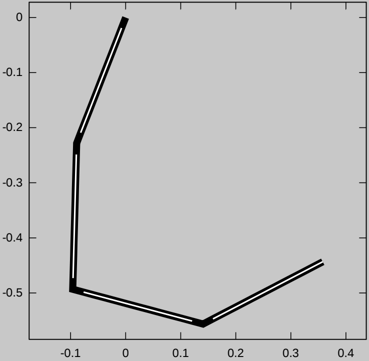
Signed Curvature

Mechanical breakdown

$$\bar{\omega}(s) = \begin{cases} 0 & s \in (0.25, 0.4) \cup (0.6, 0.75) \\ 4\pi(1 + s^2) & \text{otherwise} \end{cases}$$

Reachability

Target point $q^* = (0.35, -0.45)$



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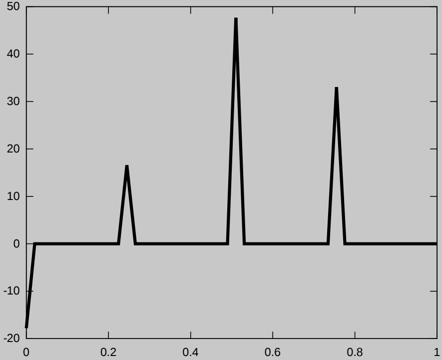


Signed Curvature

Hyper-redundant manipulator

$$\bar{\omega}(s) = \delta_0(s) + \delta_{0.25}(s) + \delta_{0.5}(s) + \delta_{0.75}(s)$$

Reachability + Obstacle Avoidance

Touch a point with the tentacle tip and minimum effort, while avoiding obstacles

Given $\Omega \subset \mathbb{R}^2$, $q^* \in \mathbb{R}^2 \setminus \Omega$, $\tau_0 > 0$, $\tau_1 > 0$ and a potential

$W_\Omega : \mathbb{R}^2 \to \mathbb{R}_0^+$ s.t. $W_\Omega(q) = 0$ for $q \in \Omega^c$ (e.g. $W_\Omega(\cdot) = \mathrm{dist}^2(\cdot, \Omega^c)$)

Minimize

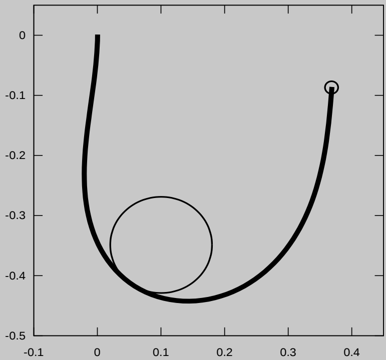
$$\frac{1}{2}\int_0^1 \frac{1}{\bar{\omega}^2}|q_{ss}|^2 ds + \frac{1}{2\tau_0}|q(1) - q^*|^2 + \frac{1}{2\tau_1}\int_0^1 W_\Omega(q(s))ds$$

subject to

$$\begin{cases} |q_{ss}| \leq \bar{\omega}\,, \quad |q_s|^2 = 1 \\ q(0) = 0\,, \quad q_s(0) = -e_2 \\ q_{ss}(1) = 0\,, \quad q_{sss}(1) = 0 \end{cases}$$

Reachability + Obstacle Avoidance

Target point $q^* = (0.37, -0.085)$, curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$



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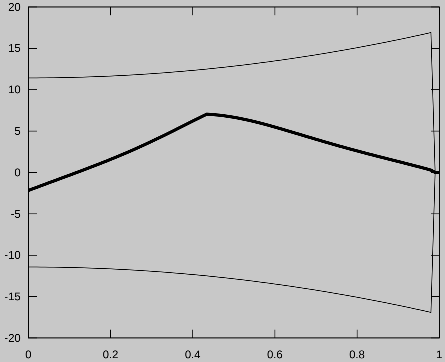


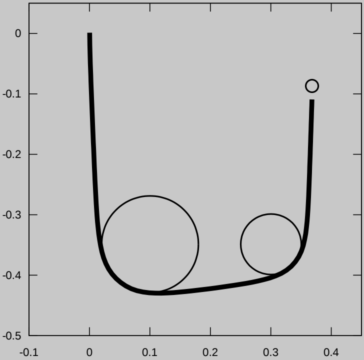
Signed Curvature

Obstacle

$\Omega = B_{0.08}(0.1, -0.35)$

Reachability + Obstacle Avoidance

Target point $q^* = (0.37, -0.085)$, curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$



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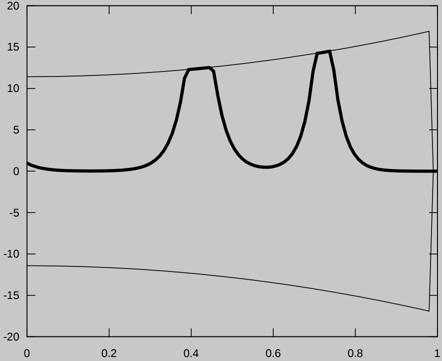


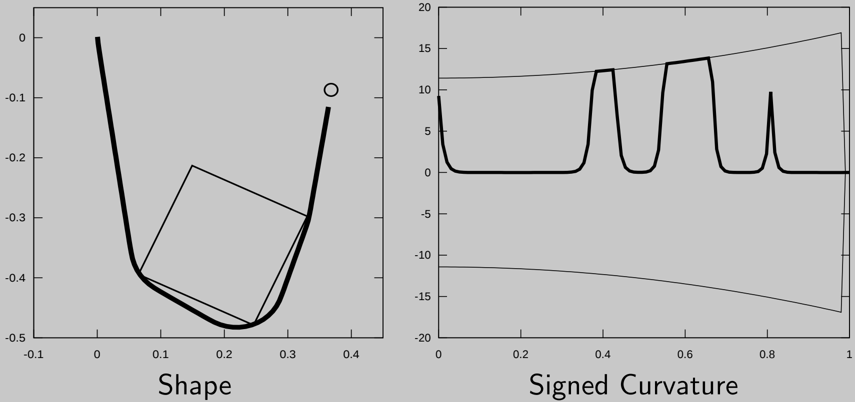
Signed Curvature

Obstacle

$$\Omega = B_{0.08}(0.1, -0.35) \cup B_{0.05}(0.3, -0.35)$$

Reachability + Obstacle Avoidance

Target point $q^* = (0.37, -0.085)$, curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$

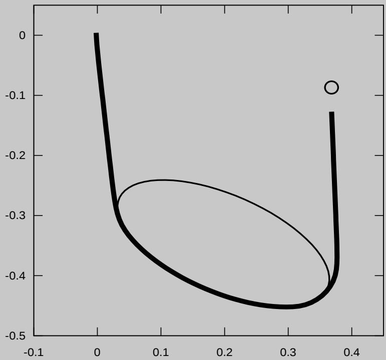


Obstacle

$\Omega = Q_{0.2}^{25^\circ}(0.2, -0.35)$

Reachability + Obstacle Avoidance

Target point $q^* = (0.37, -0.085)$, curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$



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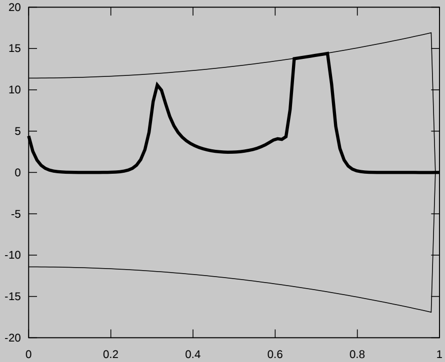


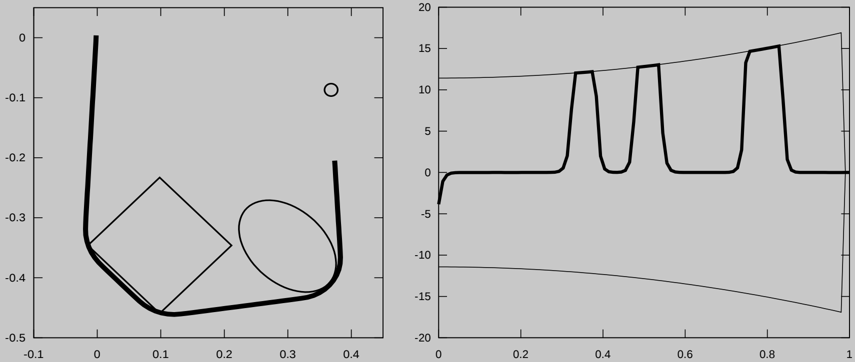
Signed Curvature

Obstacle

$$\Omega = E^{25^\circ}_{0.18, 0.08}(0.2, -0.35)$$

Reachability + Obstacle Avoidance

Target point $q^* = (0.37, -0.085)$, curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$



Shape

Signed Curvature

Obstacle

$$\Omega = Q^{45^\circ}_{0.16}(0.1, -0.35) \cup E^{45^\circ}_{0.09,0.06}(0.3, -0.35)$$

Grasping

Grasp an object with a prescribed portion of the tentacle and minimum effort

Given $\Omega \subset \mathbb{R}^2$, $\tau_0 > 0$, $\tau_1 > 0$, potentials

$W_\Omega : \mathbb{R}^2 \to \mathbb{R}_0^+$ s.t. $W_\Omega(q) = 0$ for $q \in \Omega^c$,

$W_{\partial\Omega} : \mathbb{R}^2 \to \mathbb{R}_0^+$ s.t. $W_{\partial\Omega}(q) = 0$ for $q \in \partial\Omega$,

and a function $\mu_0 : (0,1) \to \mathbb{R}_0^+$ prescribing the contact (where $\mu_0(s) > 0$)

Minimize

$$\frac{1}{2}\int_0^1 \frac{1}{\bar{\omega}^2}|q_{ss}|^2 ds + \frac{1}{2\tau_0}\int_0^1 W_\Omega(q(s))ds + \frac{1}{2\tau_1}\int_0^1 W_{\partial\Omega}(q(s))\mu_0(s)ds$$

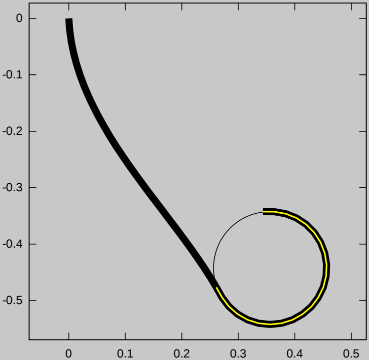
subject to

$$\begin{cases} |q_{ss}| \le \bar{\omega}\,, \quad |q_s|^2 = 1 \\ q(0) = 0\,, \quad q_s(0) = -e_2 \\ q_{ss}(1) = 0\,, \quad q_{sss}(1) = 0 \end{cases}$$

OPTIMIZATION

Grasping

Curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$



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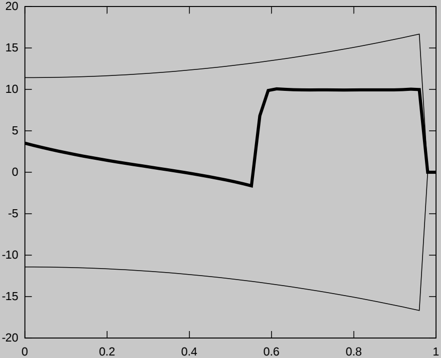


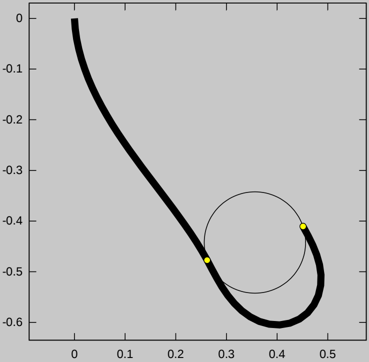
Signed Curvature

Target Object
$\Omega = B_{0.1}(0.35, -0.44)$

Contact function
$\mu(s) = \chi_{[0.55,1]}(s)$

Grasping

Curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$



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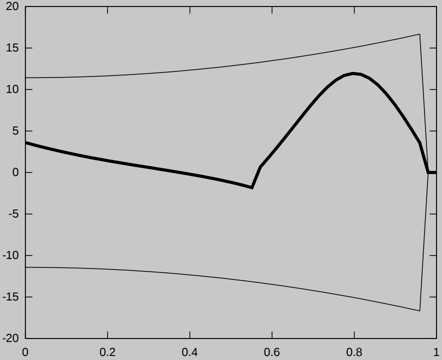


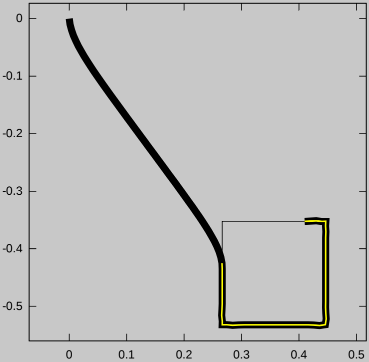
Signed Curvature

Target Object
$\Omega = B_{0.1}(0.35, -0.44)$

Contact function
$\mu(s) = \delta_{0.55}(s) + \delta_1(s)$

Grasping

No curvature bound



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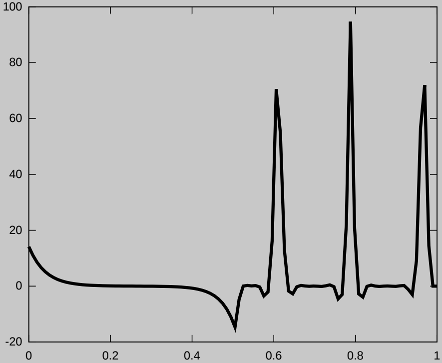


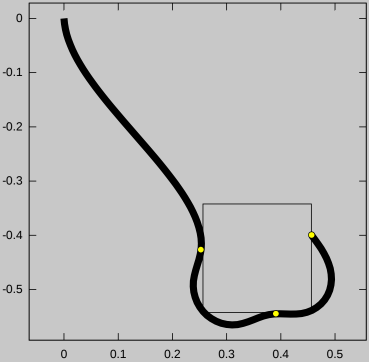
Signed Curvature

Target Object
$\Omega = Q_{0.2}(0.35, -0.44)$

Contact function
$\mu(s) = \chi_{[0.55,1]}(s)$

Grasping

Curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$



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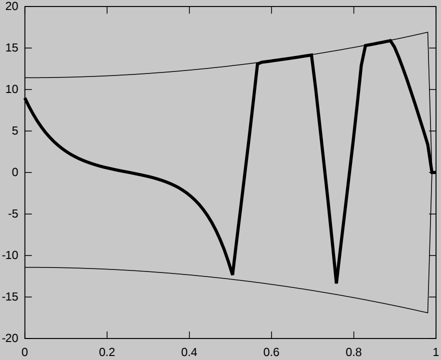


Signed Curvature

Target Object
$\Omega = Q_{0.2}(0.35, -0.44)$

Contact function
$\mu(s) = \delta_{0.55}(s) + \delta_{0.775}(s) + \delta_1(s)$

Grasping + Optimal Contact

Grasp an object at given points with minimum effort and optimal contact

Given $\Omega \subset \mathbb{R}^2$, $\tau_0 > 0$, $\tau_1 > 0$, $I_\gamma := [\gamma, 1-\gamma] \subseteq (0,1)$,

a potential $W_\Omega : \mathbb{R}^2 \to \mathbb{R}_0^+$ s.t. $W_\Omega(q) = 0$ for $q \in \Omega^c$,

and $N > 0$ fixed points $p_i \in \mathbb{R}^2$ for $i = 1, \ldots, N$ (possibly on $\partial\Omega$)

Minimize (also w.r.t. the new unknowns $s_i \in I_\gamma$, for $i = 1, \ldots, N$)

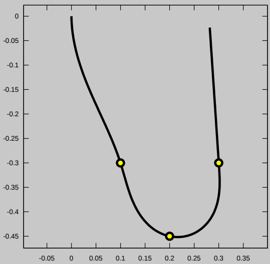
$$\frac{1}{2}\int_0^1 \frac{1}{\bar{\omega}^2}|q_{ss}|^2 ds + \frac{1}{2\tau_0}\int_0^1 W_\Omega(q(s))ds + \frac{1}{2\tau_1}\sum_{i=1}^{N}|q(s_i) - p_i|^2$$

subject to

$$\begin{cases} |q_{ss}| \leq \bar{\omega}\,, \quad |q_s|^2 = 1 \\ q(0) = 0\,, \quad q_s(0) = -e_2 \\ q_{ss}(1) = 0\,, \quad q_{sss}(1) = 0 \end{cases}$$

Grasping + Optimal Contact

Curvature bound $\bar{\omega}(s) = 4\pi(1+s^2)$



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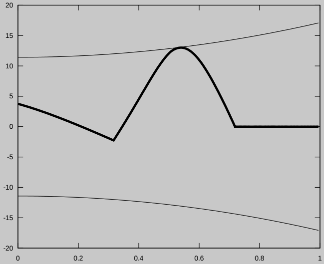
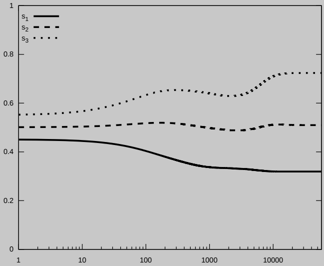


Signed Curvature



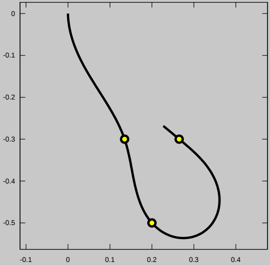
Contact optimization

No Obstacle $\Omega = \emptyset$

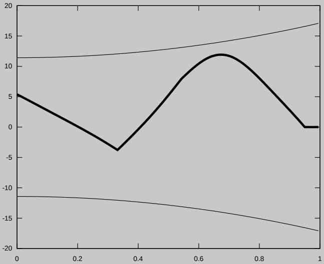
Target points: $\begin{cases} p_1 = (0.1, -0.3) \\ p_2 = (0.2, -0.45) \\ p_3 = (0.3, -0.3) \end{cases}$

Grasping + Optimal Contact

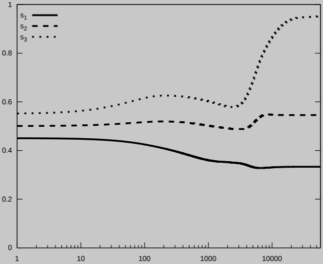
Curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$



Shape



Signed Curvature



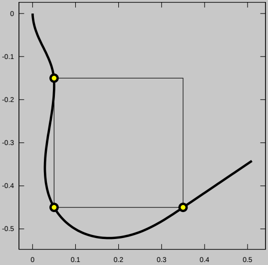
Contact optimization

No Obstacle $\Omega = \emptyset$

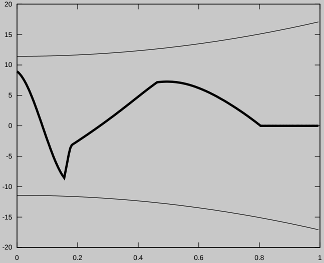
Target points: $\begin{cases} p_1 = (0.135, -0.3) \\ p_2 = (0.2, -0.5) \\ p_3 = (0.265, -0.3) \end{cases}$

Grasping + Optimal Contact

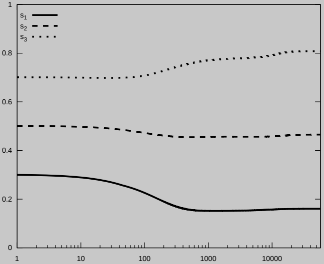
Curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$



Shape



Signed Curvature



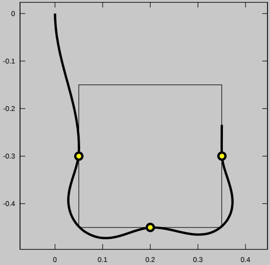
Contact optimization

Obstacle: $\Omega = Q_{0.3}(0.2, -0.3)$

$$\text{Target points: } \begin{cases} p_1 = (0.05, -0.15) \\ p_2 = (0.05, -0.45) \\ p_3 = (0.35, -0.45) \end{cases}$$

Grasping + Optimal Contact

Curvature bound $\bar{\omega}(s) = 4\pi(1 + s^2)$



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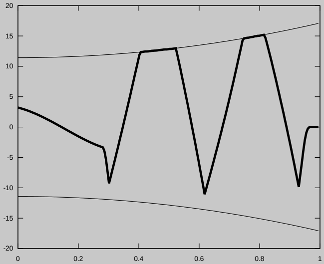
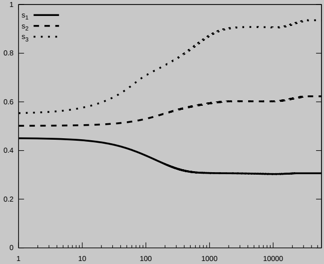


Signed Curvature



Contact optimization

Obstacle: $\Omega = Q_{0.3}(0.2, -0.3)$

Target points: $\begin{cases} p_1 = (0.05, -0.3) \\ p_2 = (0.2, -0.45) \\ p_3 = (0.35, -0.3) \end{cases}$

Optimal Grasping in Force-Closure

First-order force-closure grasp
for frictionless contact points on elliptic objects

• Geometric conditions on the contact points ensuring the immobility of the object despite external disturbances (wrenches=forces+torques).

• Contact forces should be able to generate arbitrary wrenches to counteract a disturbance wrench.

• Elliptic objects as cross sections of cylinders and ellipsoids.

Consider, in local coordinates, a generic contact point on an ellipse Ω with semi-axes $0 < b \leq a$

$$p = p(\theta) = (a\cos(\theta), b\sin(\theta)) \ , \qquad \theta \in [0, 2\pi)$$

and normal contact forces

$$f(p) = \gamma\, n(p) = -2\gamma \left(\frac{\cos(\theta)}{a}, \frac{\sin(\theta)}{b} \right) , \qquad \gamma \geq 0$$

Optimal Grasping in Force-Closure

The *wrench* associated to p is the 3D vector

$$w(p) = \begin{pmatrix} f(p) \\ p \times f(p) \end{pmatrix} = -\frac{2\gamma}{ab} \begin{pmatrix} b\cos(\theta) \\ a\sin(\theta) \\ (a^2 - b^2)\cos(\theta)\sin(\theta) \end{pmatrix}.$$

Given N contact points $\{p_1, \ldots, p_N\} \in \partial\Omega$, we say that Ω is in (first-order) *force-closure* if the set of wrenches $\{w(p_1), \ldots, w(p_N)\}$ positively spans $\mathbb{R}^3$, i.e., for all $x \in \mathbb{R}^3$, $x = \sum_{i=1}^N \alpha_i w(p_i)$ for some $\alpha_1, \ldots, \alpha_N \geq 0$.

Equivalently, for $\mathcal{W}^{FC} = (w(p_1) \ldots w(p_N)) \in \mathbb{R}^{3\times N}$

$$\text{rank } \mathcal{W}^{FC} = 3, \tag{1}$$

$$\mathcal{W}^{FC} y = 0 \text{ for some } y \in \mathbb{R}^N, \quad y_i > 0, \quad i = 1, \ldots, N. \tag{2}$$

• The full-rank condition cannot be satisfied if $a = b$ ($\Omega =$ circle).

• The number N of contact points must be equal to or greater than 4, the minimal number of generators of a three dimensional conic hull.

Optimal Grasping in Force-Closure

Take $N = 4$, $\theta_i \in [0, 2\pi)$ for $i = 1, \ldots, 4$ and set $p_i = p(\theta_i)$, $w_i = w(p_i)$.

Take $\mathcal{W} = (w_1\, w_2\, w_3) \in \mathbb{R}^{3\times 3}$ (first three columns of $\mathcal{W}^{FC}$) and

$$\mathcal{W}^{-1} = \frac{1}{\det \mathcal{W}} \begin{pmatrix} \bar{w}_1^T \\ \bar{w}_2^T \\ \bar{w}_3^T \end{pmatrix} \qquad \text{with} \quad \det \mathcal{W} = w_1 \cdot w_2 \times w_3,$$

whose rows, for $i = 1, 2, 3$, are given by the components of the vectors (cycling indices notation!)

$$\bar{w}_i := \begin{pmatrix} -a(a^2 - b^2)\sin(\theta_{i+1})\sin(\theta_{i+2})(\cos(\theta_{i+1}) - \cos(\theta_{i+2})) \\ b(a^2 - b^2)\cos(\theta_{i+1})\cos(\theta_{i+2})(\sin(\theta_{i+1}) - \sin(\theta_{i+2})) \\ -ab\sin(\theta_{i+1} - \theta_{i+2}) \end{pmatrix}$$

Then Ω is in *force-closure* if and only if

$$\det \mathcal{W} \neq 0\,, \qquad \operatorname{sign}(\det \mathcal{W})\, \bar{w}_i \cdot w_4 \leq 0\,, \quad i = 1, 2, 3\,.$$

Optimal Grasping in Force-Closure

Set

$$\varepsilon_0 = \varepsilon_0(a,b) := \varepsilon \max_{\theta_1,\theta_2,\theta_3} |\det \mathcal{W}(\theta_1,\theta_2,\theta_3)| \qquad \text{for } \varepsilon \in (0,1)$$

$$\Theta = \{\theta_1,\theta_2,\theta_3,\theta_4\} \subset [0,2\pi)$$

and define

$$F(\Theta) = \frac{1}{2}\sum_{i=0}^{3} F_i(\Theta),$$

with

$$F_0(\Theta) = \max^2\{0, \varepsilon_0 - |\det \mathcal{W}|\},$$

$$F_i(\Theta) = \begin{cases} \max^2\{0, \bar{w}_i \cdot w_4\} & \text{if } \det \mathcal{W} > 0, \\ \min^2\{0, \bar{w}_i \cdot w_4\} & \text{otherwise,} \end{cases} \qquad i = 1,2,3$$

- F is a non negative by construction
- Θ is an absolute minimizer of F, achieving $F(\Theta) = 0$, if and only if Ω is in force-closure and $|\det \mathcal{W}| \geq \varepsilon_0$.

Optimal Grasping in Force-Closure

Grasp an ellipse in force-closure with minimum effort and optimal contact

Given an ellipse $\Omega \subset \mathbb{R}^2$, $\tau_0 > 0$, $\tau_1 > 0$, $\tau_2 > 0$, $I_\gamma := [\gamma, 1-\gamma] \subseteq (0,1)$,

a potential $W_\Omega : \mathbb{R}^2 \to \mathbb{R}_0^+$ s.t. $W_\Omega(q) = 0$ for $q \in \Omega^c$,

Minimize (w.r.t. $s_i \in I_\gamma$ and also $\theta_i \in [0, 2\pi)$, for $i = 1, \ldots, 4$)

$$\frac{1}{2}\int_0^1 \frac{1}{\bar{\omega}^2}|q_{ss}|^2 ds + \frac{1}{2\tau_0}\int_0^1 W_\Omega(q(s))ds + \frac{1}{2\tau_1}\sum_{i=1}^{4}|q(s_i) - p(\theta_i)|^2 + \frac{1}{2\tau_2}F(\Theta)$$

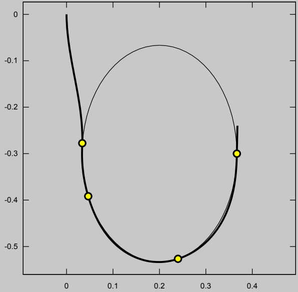
subject to

$$\begin{cases} |q_{ss}| \leq \bar{\omega}\,, \quad |q_s|^2 = 1 \\ q(0) = 0\,, \quad q_s(0) = -e_2 \\ q_{ss}(1) = 0\,, \quad q_{sss}(1) = 0 \end{cases}$$

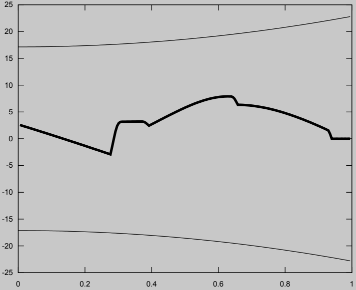
- Optimization in Θ is unconstrained due to periodicity!

Optimal Grasping in Force-Closure

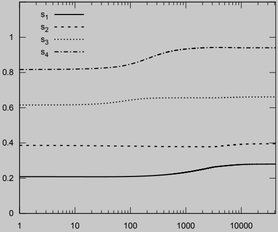
Ω = ellipse with semi-axes $(a, b) = (0.16, 0.23)$



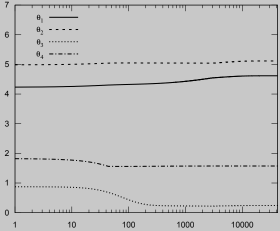
Shape



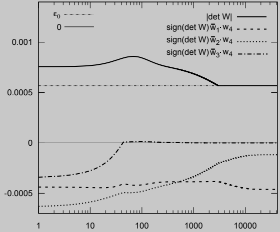
Signed Curvature



Contact Points



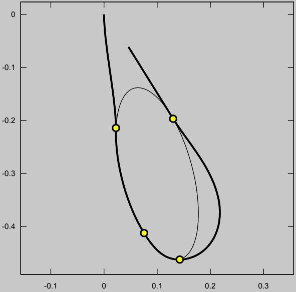
Target Points



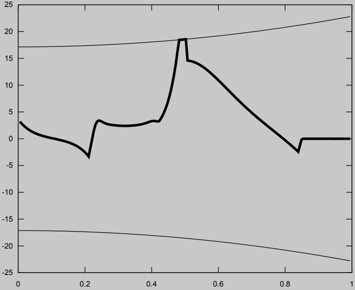
Force-Closure

Optimal Grasping in Force-Closure

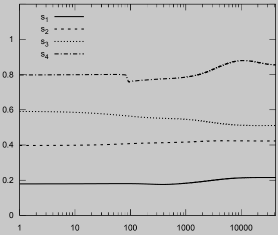
Ω = ellipse with semi-axes $(a, b) = (0.06, 0.16)$ rotated by 15°



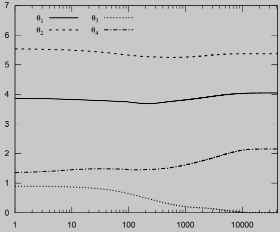
Shape



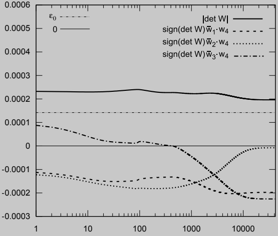
Signed Curvature



Contact Points



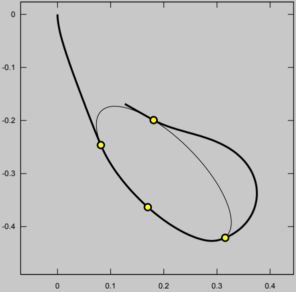
Target Points



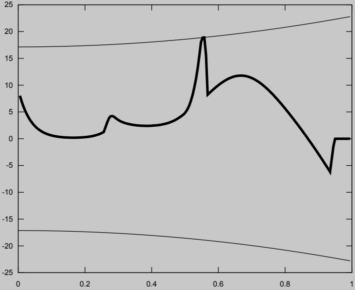
Force-Closure

Optimal Grasping in Force-Closure

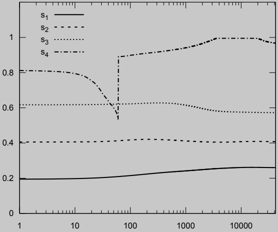
Ω = ellipse with semi-axes $(a, b) = (0.06, 0.16)$ rotated by 45°



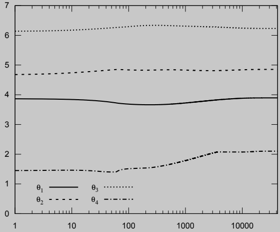
Shape



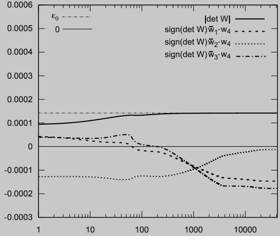
Signed Curvature



Contact Points



Target Points



Force-Closure

Dynamic Reachability

Touch a point with the tentacle tip and stop with minimum effort

Given $T > 0$, $q^* \in \mathbb{R}^2$ and $\tau_0 > 0$

Minimize $$\frac{1}{2}\int_0^T\int_0^1 u^2 ds\, dt + \frac{1}{2\tau_0}\int_0^T |q(1,t) - q^*|^2 dt + \frac{1}{2}\int_0^1 |q_t(s,T)|^2 ds$$

subject to

$$\begin{cases} \rho q_{tt} = \left[\sigma q_s - H q_{ss}^{\perp}\right]_s - \left[G q_{ss} + H q_s^{\perp}\right]_{ss} & \text{in } (0,1)\times(0,T) \\ |q_s|^2 = 1 & \text{in } (0,1)\times(0,T) \\ |u| \le 1 & \text{in } (0,1)\times(0,T) \\ q(0,t) = 0 & t \in (0,T) \\ q_s(0,t) = -e_2 & t \in (0,T) \\ q_{ss}(1,t) = 0 & t \in (0,T) \\ q_{sss}(1,t) = 0 & t \in (0,T) \\ \sigma(1,t) = 0 & t \in (0,T) \\ q(s,0) = q^0(s) & s \in (0,1) \\ q_t(s,0) = q^1(s) & s \in (0,1) \end{cases}$$

Dynamic Reachability

Introduce the adjoint state $(\bar{q}, \bar{\sigma})$ and form the Lagrangian

$$\mathcal{L} = \frac{1}{2}\int_0^T\int_0^1 u^2 ds\, dt + \frac{1}{2\tau_0}\int_0^T |q(1,t) - q^*|^2 dt + \frac{1}{2}\int_0^1 \rho(s)|q_t(s,T)|^2 ds$$

$$+ \int_0^T\int_0^1 \bar{q}\cdot\left(\rho q_{tt} - \left[\sigma q_s - Hq_{ss}^{\perp}\right]_s + \left[Gq_{ss} + Hq_s^{\perp}\right]_{ss}\right) ds\, dt$$

$$+\frac{1}{2}\int_0^T\int_0^1 \bar{\sigma}(|q_s|^2 - 1) ds\, dt$$

Take admissible variations and impose optimality.

After (very long) integration by parts get the optimality system:
find (q, σ), $(\bar{q}, \bar{\sigma})$ and $u \in [-1, 1]$ such that for $(s, t) \in (0, 1) \times (0, T)$...

Dynamic Reachability

$$\begin{cases} \rho q_{tt} = \left[\sigma q_s - H q_{ss}^{\perp}\right]_s \\ \qquad - \left[G q_{ss} + H q_s^{\perp}\right]_{ss} \\ |q_s|^2 = 1 \\ q(0,t) = 0, \quad q_s(0,t) = -e_2 \\ q_{ss}(1,t) = 0, \quad q_{sss}(1,t) = 0 \\ \sigma(1,t) = 0 \\ q(s,0) = q^0(s) \\ q_t(s,0) = q^1(s) \end{cases}$$

$$\begin{cases} \rho \bar{q}_{tt} = \left[\sigma \bar{q}_s - H \bar{q}_{ss}^{\perp} + \bar{\sigma} q_s + h q_{ss}^{\perp}\right]_s \\ \qquad - \left[G \bar{q}_{ss} + H \bar{q}_s^{\perp} + g q_{ss} - h q_s^{\perp}\right]_{ss} \\ \bar{q}_s \cdot q_s = 0 \\ \bar{q}(0,t) = 0, \quad \bar{q}_s(0,t) = 0 \\ \bar{q}_{ss}(1,t) = 0 \\ \bar{q}_{sss}(1,t) = -\frac{1}{\varepsilon}\left(\bar{\sigma} q_s + \frac{1}{\tau_0}(q - q^*)\right)(1,t) \\ \bar{\sigma}(1,t) = -\frac{1}{\tau_0}(q - q^*) \cdot q_s(1,t) \\ \bar{q}(s,T) = -q_t(s,T) \\ \bar{q}_t(s,T) = 0 \end{cases}$$

$$\int_0^T \int_0^1 \Big(u - \omega h(q, \bar{q}, \mu)\Big)(v - u)\, ds\, dt \geq 0 \quad \forall\, v \in [-1, 1]$$

where

$$G(q, \nu, \varepsilon, \omega) = \varepsilon + \nu\left(|q_{ss}|^2 - \omega^2\right)_+ \qquad H(q, \mu, u, \omega) = \mu\left(\omega u - q_s \times q_{ss}\right)$$

$$g(q, \nu, \omega) = 2\nu \chi_{[0,+\infty)}\left(|q_{ss}|^2 - \omega^2\right) \qquad h(q, \bar{q}, \mu) = \mu\left(\bar{q}_s \times q_{ss} + q_s \times \bar{q}_{ss}\right)$$

Ongoing Work & Future Developments

- Unscrew the cap of a jar: dynamic interaction (friction)
- Dynamic pathfinding and grasping
- Two-way interactions with fluids (Lattice-Boltzmann)
- Multiple tentacles in cooperation
- Extension to a 3D model (torsion, self-collisions, ...)

Ongoing Work & Future Developments

- Unscrew the cap of a jar: dynamic interaction (friction)
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THANK YOU FOR YOUR ATTENTION!

Controlled Dynamics - Numerical Approximation

Finite Difference (D_+, D_-, D_c^2) + Verlet Velocity (Leapfrog)

Discretization:

space $[0,1]$, $\Delta s = 1/N,\ s_k = k\Delta s,\ \ k = 0, ..., N$

time $[0,T]$, $\Delta t = T/M,\ t_n = n\Delta t,\ \ n = 0, ..., M$

$\Longrightarrow \chi(s_k, t_n) \approx \chi_k^n$

Initial conditions:

position $q_k^0 = q^0(s_k)$

velocity $v_k^0 = q^1(s_k)$

Boundary conditions (2 ghost nodes):

anchor point	$q_0^n = 0$
fixed tangent	$q_{-1}^n = q_0^n + e_2 \Delta s$
zero bending moment	$q_{N+1}^n - 2q_N^n + q_{N-1}^n = 0$
zero shear stress	$q_{N+1}^n - 3q_N^n + 3q_{N-1}^n - q_{N-2}^n = 0$
zero tension	$\sigma_N^n = 0$

Controlled Dynamics - Numerical Approximation

Finite Difference (D_+, D_-, D_c^2) + Verlet Velocity (Leapfrog)

Constraints:

curvature $\quad G_k^n = \varepsilon_k + \nu_k \left(|D_c^2 q_k^n|^2 - \omega_k^2 \right)_+$

control $\quad H_k^n = \mu_k \left(\omega_k u_k^n - D_- q_k^n \times D_c^2 q_k^n \right)$

Accelerations:

$$a(q_k^n, \sigma_k^n) = \frac{1}{\rho_k} \left(D_+ \left(\sigma_k^n D_- q_k^n - H_k^n D_c^2 q_k^{n\perp} \right) - D_c^2 \left(G_k^n D_c^2 q_k^n + H_k^n D_- q_k^{n\perp} \right) \right)$$

Nonlinear system:

$$\begin{cases} q_k^{n+1} = q_k^n + v_k^n + \frac{1}{2} a(q_k^n, \sigma_k^n) \Delta t^2 \\ |D_- q_k^{n+1}|^2 = 1 \end{cases}$$

Newton method for (q_k^{n+1}, σ_k^n)

$k = 1, ..., N-1$

Velocities:

$$v_k^{n+1} = v_k^n + \frac{1}{2} (a(q_k^n, \sigma_k^n) + a(q_k^{n+1}, \sigma_k^n)) \Delta t$$

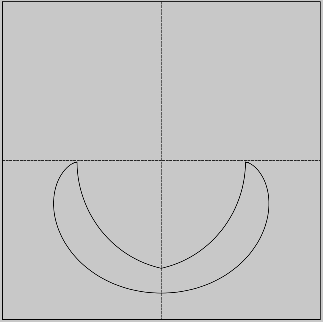
BACK

Reachable Set

$$\mathcal{R} = \{q^* \in \mathbb{R}^2 \mid q^* = q(1) \quad \text{where } q \text{ is an equilibrium}\}$$

$$\bar{\omega} \equiv const \implies \text{for } \ell, r \in [-1,1],\ \alpha \in [0,1],\ u_\alpha^{\ell,r}(s) := \begin{cases} \ell & 0 \leq s < \alpha \\ r & \alpha \leq s \leq 1 \end{cases}$$

$$\partial\mathcal{R} = \left\{ q(1) \mid q_s \times q_{ss} = \bar{\omega} u_\alpha,\ \alpha \in [0,1],\ u_\alpha \in \left\{ u_\alpha^{1,0}, u_\alpha^{-1,0}, u_\alpha^{-1,1}, u_\alpha^{1,-1} \right\} \right\}$$



$\bar{\omega} = \pi$

Reachable Set

$$\mathcal{R} = \{q^* \in \mathbb{R}^2 \mid q^* = q(1) \quad \text{where } q \text{ is an equilibrium}\}$$

$$\bar{\omega} \equiv const \implies \text{for } \ell, r \in [-1,1],\ \alpha \in [0,1],\ u_\alpha^{\ell,r}(s) := \begin{cases} \ell & 0 \le s < \alpha \\ r & \alpha \le s \le 1 \end{cases}$$

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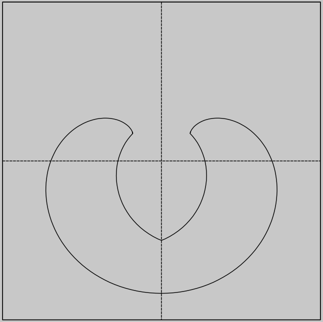


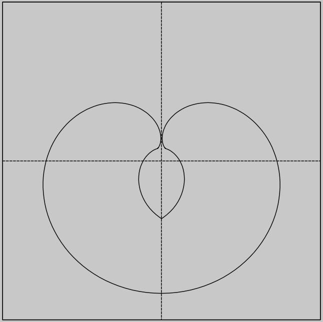
$\bar{\omega} = \frac{3}{2}\pi$

Reachable Set

$$\mathcal{R} = \{q^* \in \mathbb{R}^2 \mid q^* = q(1) \quad \text{where } q \text{ is an equilibrium}\}$$

$$\bar{\omega} \equiv const \implies \text{for } \ell, r \in [-1, 1],\ \alpha \in [0, 1],\ u_\alpha^{\ell,r}(s) := \begin{cases} \ell & 0 \le s < \alpha \\ r & \alpha \le s \le 1 \end{cases}$$

$$\partial\mathcal{R} = \left\{ q(1) \mid q_s \times q_{ss} = \bar{\omega} u_\alpha,\ \alpha \in [0, 1],\ u_\alpha \in \left\{ u_\alpha^{1,0}, u_\alpha^{-1,0}, u_\alpha^{-1,1}, u_\alpha^{1,-1} \right\} \right\}$$



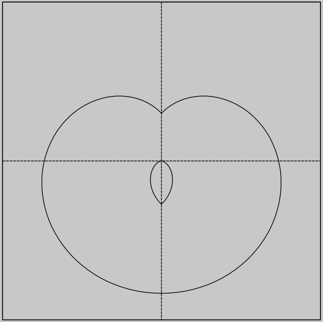
$\bar{\omega} = \frac{3}{2}\pi + 1$

Reachable Set

$$\mathcal{R} = \{q^* \in \mathbb{R}^2 \mid q^* = q(1) \quad \text{where } q \text{ is an equilibrium}\}$$

$$\bar{\omega} \equiv const \implies \text{for } \ell, r \in [-1,1],\ \alpha \in [0,1],\ u_\alpha^{\ell,r}(s) := \begin{cases} \ell & 0 \le s < \alpha \\ r & \alpha \le s \le 1 \end{cases}$$

$$\partial\mathcal{R} = \left\{ q(1) \mid q_s \times q_{ss} = \bar{\omega} u_\alpha,\ \alpha \in [0,1],\ u_\alpha \in \left\{u_\alpha^{1,0}, u_\alpha^{-1,0}, u_\alpha^{-1,1}, u_\alpha^{1,-1}\right\} \right\}$$



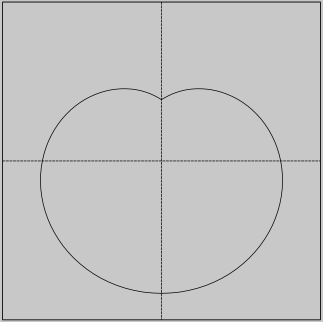
$\bar{\omega} = 2\pi$

Reachable Set

$$\mathcal{R} = \{q^* \in \mathbb{R}^2 \mid q^* = q(1) \quad \text{where } q \text{ is an equilibrium}\}$$

$$\bar{\omega} \equiv const \implies \text{for } \ell, r \in [-1,1],\ \alpha \in [0,1],\ u_\alpha^{\ell,r}(s) := \begin{cases} \ell & 0 \le s < \alpha \\ r & \alpha \le s \le 1 \end{cases}$$

$$\partial\mathcal{R} = \left\{ q(1) \mid q_s \times q_{ss} = \bar{\omega} u_\alpha,\ \alpha \in [0,1],\ u_\alpha \in \left\{ u_\alpha^{1,0}, u_\alpha^{-1,0}, u_\alpha^{-1,1}, u_\alpha^{1,-1} \right\} \right\}$$



$\bar{\omega} = \frac{9}{4}\pi$

BACK

Reachability: Optimization

Inextensibility constraint: Lagrange multiplier σ

Curvature constraint: Slack variable z

Augmented Lagrangian (multiplier λ and penalty parameter $\rho_\lambda > 0$):

$$\mathcal{L}(q,\sigma,z,\lambda) = \frac{1}{2}\int_0^1 \frac{1}{\bar{\omega}^2}|q_{ss}|^2 ds + \frac{1}{2\tau_0}|q(1) - q^*|^2 + \frac{1}{2}\int_0^1 \sigma(|q_s|^2 - 1) ds$$

$$+\frac{1}{2}\int_0^1 \lambda(|q_{ss}|^2 - \bar{\omega}^2 + z) ds + \frac{1}{4\rho_\lambda}\int_0^1 (|q_{ss}|^2 - \bar{\omega}^2 + z)^2 ds$$

Method of Multipliers: given $\lambda^{(0)}$ iterate on $i \geq 0$ up to convergence

$$\begin{cases} (q^{(i+1)}, \sigma^{(i+1)}, z^{(i+1)}) = \underset{q,\sigma,\; z\geq 0}{\operatorname{argmin}} \mathcal{L}(q,\sigma,z,\lambda^{(i)}) \\ \\ \lambda^{(i+1)} = \lambda^{(i)} + \dfrac{1}{\rho_\lambda}(|q_{ss}^{(i+1)}|^2 - \bar{\omega}^2 + z^{(i+1)}) \end{cases}$$

Reachability: Optimization

Subproblem $\operatorname{argmin}_{q,\sigma,\ z\geq 0} \mathcal{L}(q,\sigma,z,\lambda^{(i)})$

Take admissible variations and impose optimality:

$$< \delta_\sigma \mathcal{L}, \chi >= \int_0^1 (|q_s|^2 - 1)\chi ds = 0 \quad \forall\, \chi$$

$$\implies \quad |q_s|^2 = 1 \quad \text{a.e. in } (0,1)$$

$$< \delta_z \mathcal{L}, v >= \frac{1}{2}\int_0^1 \left(\lambda^{(i)} + \frac{1}{\rho_\lambda}(|q_{ss}|^2 - \bar{\omega}^2 + z) \right)(v - z) ds \geq 0 \quad \forall\, v \geq 0$$

$$\implies z = \max\left\{ -\lambda^{(i)}\rho_\lambda - |q_{ss}|^2 + \bar{\omega}^2, 0 \right\} \quad \text{a.e. in } (0,1)$$

Reachability: Optimization

Subproblem $\underset{q,\sigma,\ z\geq 0}{\operatorname{argmin}} \mathcal{L}(q, \sigma, z, \lambda^{(i)})$

Take admissible variations and impose optimality:

$$< \delta_q \mathcal{L}, w > = \int_0^1 \Big(\Lambda(q_{ss}, \lambda^{(i)}) q_{ss} \cdot w_{ss} + \sigma q_s \cdot w_s \Big) ds + \frac{1}{\tau_0} (q(1) - q^*) \cdot w(1) = 0$$

$$\text{with} \qquad \Lambda(q_{ss}, \lambda^{(i)}) = \frac{1}{\bar{\omega}^2} + \max \left\{ \lambda^{(i)} + \frac{1}{\rho_\lambda} (|q_{ss}|^2 - \bar{\omega}^2), 0 \right\}$$

Integrate by parts and impose boundary conditions:

$$\begin{cases} \left[\Lambda(q_{ss}, \lambda^{(i)}) q_{ss} \right]_{ss} - [\sigma q_s]_s = 0 & \text{in } (0,1) \\ |q_s|^2 = 1 & \text{in } (0,1) \\ q(0) = 0, \quad q_s(0) = -e_2 & \\ q_{ss}(1) = 0, \quad q_{sss}(1) = 0 & \\ \sigma(1) q_s(1) + \frac{1}{\tau_0}(q(1) - q^*) = 0 & \end{cases} \implies \quad (q^{(i+1)}, \sigma^{(i+1)})$$

Reachability - Numerical Approximation

Method of Multipliers: given $\lambda^{(0)}$ iterate on $i \geq 0$ up to convergence

- Discretization:

$$
\begin{cases}
D_c^2 \left(\Lambda(D_c^2 q_k, \lambda_k^{(i)}) D_c^2 q_k \right) - D_+ (\sigma_k D_- q_k) = 0 & k = 1, ..., N-1 \\
|D_- q_k|^2 = 1 & k = 1, ..., N-1 \\
q_0 = 0, \quad q_{-1}^n = q_0^n + e_2 \Delta s \\
q_{N+1}^n - 2q_N^n + q_{N-1}^n = 0 \\
q_{N+1}^n - 3q_N^n + 3q_{N-1}^n - q_{N-2}^n = 0 \\
\sigma_N D_- q_N + \frac{1}{\tau_0}(q_N - q^*) = 0
\end{cases}
$$

- Solution $(q^{(i+1)}, \sigma^{(i+1)})$ via Quasi-Newton method (freezing q in Λ)

- Update multipliers:

$$
\lambda_k^{(i+1)} = \max \left\{ \lambda_k^{(i)} + \frac{1}{\rho_\lambda} (|D_c^2 q_k^{(i+1)}|^2 - \bar{\omega}_k^2), 0 \right\}
$$

BACK

Reachability + Obstacle Avoidance: Optimization

Method of Multipliers: given $\lambda^{(0)}$ iterate on $i \geq 0$ up to convergence

$$\begin{cases} \left[\Lambda(q_{ss}, \lambda^{(i)}) q_{ss}\right]_{ss} - [\sigma q_s]_s + \frac{1}{\tau_1} \nabla W_\Omega(q(s)) = 0 & \text{in } (0,1) \\ |q_s|^2 = 1 & \text{in } (0,1) \\ q(0) = 0, \quad q_s(0) = -e_2 & \\ q_{ss}(1) = 0, \quad q_{sss}(1) = 0 & \\ \sigma(1) q_s(1) + \frac{1}{\tau_0}(q(1) - q^*) = 0 & \Longrightarrow \quad (q^{(i+1)}, \sigma^{(i+1)}) \end{cases}$$

$$\Lambda(q_{ss}, \lambda^{(i)}) = \frac{1}{\bar{\omega}^2} + \max\left\{ \lambda^{(i)} + \frac{1}{\rho_\lambda}(|q_{ss}|^2 - \bar{\omega}^2), 0 \right\}$$

$$\lambda^{(i+1)} = \max\left\{ \lambda^{(i)} + \frac{1}{\rho_\lambda}(|q_{ss}^{(i+1)}|^2 - \bar{\omega}^2), 0 \right\}$$

BACK

Grasping: Optimization

Method of Multipliers: given $\lambda^{(0)}$ iterate on $i \geq 0$ up to convergence

$$
\begin{cases}
\left[\Lambda(q_{ss},\lambda^{(i)})q_{ss}\right]_{ss} - [\sigma q_s]_s & \\
+\frac{1}{\tau_0}\nabla W_\Omega(q(s)) + \frac{1}{\tau_1}\nabla W_{\partial\Omega}(q(s))\mu_0(s) = 0 & \text{in } (0,1) \\
& \\
|q_s|^2 = 1 & \text{in } (0,1) \\
q(0) = 0, \quad q_s(0) = -e_2 & \\
q_{ss}(1) = 0, \quad q_{sss}(1) = 0 & \\
\sigma(1) = 0 & \Longrightarrow \quad (q^{(i+1)}, \sigma^{(i+1)})
\end{cases}
$$

$$
\Lambda(q_{ss},\lambda^{(i)}) = \frac{1}{\bar{\omega}^2} + \max\left\{\lambda^{(i)} + \frac{1}{\rho_\lambda}(|q_{ss}|^2 - \bar{\omega}^2), 0\right\}
$$

$$
\lambda^{(i+1)} = \max\left\{\lambda^{(i)} + \frac{1}{\rho_\lambda}(|q_{ss}^{(i+1)}|^2 - \bar{\omega}^2), 0\right\}
$$

BACK

Grasping + Optimal Contact: Optimization

Method of Multipliers: given $\lambda^{(0)}$ iterate on $i \geq 0$ up to convergence

$$\begin{cases} \left[\Lambda(q_{ss},\lambda^{(i)})q_{ss}\right]_{ss} - [\sigma q_s]_s & \\ +\frac{1}{\tau_0}\nabla W_\Omega(q(s)) + \frac{1}{\tau_1}\sum_{i=1}^N (q(s)-p_i)\delta_{s_i}(s) = 0 & \text{in } (0,1) \\ |q_s|^2 = 1 & \text{in } (0,1) \\ \frac{1}{\tau_1}(q(s_i)-p_i)\cdot q_s(s_i)(w_i - s_i) \geq 0\,, & \forall w_i \in I_\gamma \\ & i = 1,\dots,N \\ q(0)=0, \quad q_s(0) = -e_2 & \\ q_{ss}(1)=0, \quad q_{sss}(1)=0 & \\ \sigma(1) = 0 & \Longrightarrow \quad (q^{(i+1)},\sigma^{(i+1)}) \end{cases}$$

$$\Lambda(q_{ss},\lambda^{(i)}) = \frac{1}{\bar{\omega}^2} + \max\left\{\lambda^{(i)} + \frac{1}{\rho_\lambda}(|q_{ss}|^2 - \bar{\omega}^2), 0\right\}$$

$$\lambda^{(i+1)} = \max\left\{\lambda^{(i)} + \frac{1}{\rho_\lambda}(|q_{ss}^{(i+1)}|^2 - \bar{\omega}^2), 0\right\}$$

Optimal Grasping in Force-Closure: Optimization

Method of Multipliers: given $\lambda^{(0)}$ iterate on $i \geq 0$ up to convergence

$$\begin{cases} \left[\Lambda(q_{ss},\lambda^{(i)})q_{ss}\right]_{ss} - [\sigma q_s]_s \\ +\frac{1}{\tau_0}\nabla W_\Omega(q(s)) + \frac{1}{\tau_1}\sum_{i=1}^4 (q(s)-p(\theta_i))\delta_{s_i}(s) = 0 & \text{in } (0,1) \\ |q_s|^2 = 1 & \text{in } (0,1) \\ \frac{1}{\tau_1}(q(s_i)-p(\theta_i))\cdot q_s(s_i)(w_i - s_i) \geq 0\,, & \forall w_i \in I_\gamma,\ i=1,\dots,4 \\ \frac{1}{\tau_1}(q(s_i)-p(\theta_i))\cdot \frac{\partial}{\partial\theta_i}p(\theta_i) + \frac{1}{\tau_2}\frac{\partial}{\partial\theta_i}F(\Theta) = 0 & i=1,\dots,4 \\ q(0)=0,\ q_s(0)=-e_2,\ q_{ss}(1)=0,\ q_{sss}(1)=0 \\ \sigma(1)=0 & \Longrightarrow \quad (q^{(i+1)},\sigma^{(i+1)}) \end{cases}$$

$$\Lambda(q_{ss},\lambda^{(i)}) = \frac{1}{\bar\omega^2} + \max\left\{\lambda^{(i)} + \frac{1}{\rho_\lambda}(|q_{ss}|^2 - \bar\omega^2), 0\right\}$$

$$\lambda^{(i+1)} = \max\left\{\lambda^{(i)} + \frac{1}{\rho_\lambda}(|q_{ss}^{(i+1)}|^2 - \bar\omega^2), 0\right\}$$

BACK

Dynamic Reachability - Numerical Approximation

Discretization: Finite Difference (D_+, D_-, D_c^2) + Verlet Velocity (Leapfrog)

Adjoint-based Projected Gradient Descent Method:

given $u^{(0)}$ (e.g. Static Control) iterate on $i \geq 0$ up to convergence

- Solve the forward system with fixed $u^{(i)} \quad \Longrightarrow (q^{(i)}, \sigma^{(i)})$
- Solve the backward system with fixed $u^{(i)}$, $q^{(i)}$, $\sigma^{(i)} \quad \Longrightarrow (\bar{q}^{(i)}, \bar{\sigma}^{(i)})$
- Update and project the control (for a suitable step α)

$$u^{(i+1)} = \mathbb{P}_{[-1,1]} \left\{ u^{(i)} - \alpha \left(u^{(i)} - \omega h(q^{(i)}, \bar{q}^{(i)}, \mu) \right) \right\}$$

BACK